FINDING THE LAST DIGIT OF ANY NUMBER TO ANY POWER (CYCLICITY)

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Before we go into the topic, let's see the power of all the numbers from 1 to 9.

Note that we can find some pattern in their units digit after a fixed cycle..

NOTE: -- is used wherever the rest of the number is not of concern

CYCLE -- It is the units digit cycle.

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<th>N</th>
<th>N^2</th>
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Here we see that in all of the numbers from 1 to 9 the pattern in powers repeats after every cycle. This is what we can use to find the last digit of any number to any power.

Lets start with an easy example to explain the basic principles and concept behind what is done.
1) Suppose we to find the last digit of $733^{34}$

**STEP 1**

If we think about this question we can see that in 733 we are only concerned with the units digit so the tens hundreds and so on digits have no role in our working. Thus 475450453 or 342343 or 3 does not change our answer because the units digit of any power is only dependent on the units digit of the number we are taking. So now the question reduces to-

What is the last digit of $3^{34}$?

**STEP 2**

We have seen in the table above that the pattern of 3 is 3, 9, 7, 1 so it repeats after every cycle of 4. Suppose the question was thus in a more mathematical form we can write:

\[
\begin{align*}
3^{(4k+1)} &= -3 \\
3^{(4k+2)} &= -9 \\
3^{(4k+3)} &= -7 \\
3^{(4k+4)} &= -1
\end{align*}
\]

This means that if we find the remainder of the power from 4 we can find the last digit.

Now since,

\[
3^{34} = 3^{(4\times 8 + 2)}
\]

this matches out form $3^{(4k + 2)}$

\[
3^{34} \Rightarrow 3^2 = 9
\]

So the last digit of the Question is 9.

2) Find last digit of $4324836^{42314234}$

Just by looking (Vilokanan Sutra) we can say that the units digit is same as $6^{42314234}$

pattern of 6 is all 6's so we can simply say units digit is 6. Done.
3) Find the units digit of $4377^{5797}$

**STEP 1**

Just like before –

$7^{5797}$

**STEP 2**

Find remainder of 5797 from 4 we get 1

Answer is same as $7^1 = 7$

Units Digit is 7

4) Units digit of $523^{5245}$

**S1** (Step 1) - $3^{5245}$

**S2** - $5245 / 4$ leaves Rem 1

$3^1 = 3$

Units digit is 3.

5) Units Digit of $4234^{47632}$

**S1** - $4^{47632}$

**S2** – Rem of 47632 from 4 = 0

Don’t mistake this as $4^0$ because this is wrong! Instead we write $4^4$ because it matches the type $4^{4k+4} = -6$

So our answer is 6 not 1.

6) Units digit of $524567^{2545345543543545}$

(I have purposely taken a big power to show how easy it can be!!)
– As we did in the last Questions we write it as

\[72545345543543545\]

In the last question I had told to find the remainder from 4. But finding remainder from 4 in this example is no easy task. Here is where osculation comes handy. I am not going into osculation because Swami Bharati Krishna Tirthaji Maharaja has already done a wonderful and elaborate work on it in his book (See reference).

**RULE**– ANY number divided by 4 will leave same remainder as the last two digits divided by 4.

Here is what i am saying-

2545345543543545 leaves some remainder when divided by 4. However this remainder is same as that of 45 when divided by 4 which can easily be found to be 1.

So, \(7^1 = 7\)

UD (Units digit) is 7.

7) UD of 84732423059 9808907789088

Don’t be afraid of big numbers because they don’t mean anything to us!!

S1 - 9808907789088

S2 - 9808907789088 when divided by 4 leaves same Rem as 88 div by 4.

\(9^4 = \_1\)

UD is 1.

8) Find 40234986783 42344353452345 \(\mod 10\)

Easy, or Hard? You decide.

S1 - 342344353452345

S2 - \(3^{45} \to 3^1\)

UD is 3
-- Any number to any power is no problem. We are just concerned with the last digit of the number and last two digits of the power

-- RULE– ANY number divided by 4 will leave same remainder as the last two digits divided by 4.

-- If the power is divisible by 4 we replace the power by 4.
For example

\[ 3^{24} \Rightarrow 3^{4} \]
\[ 7^{28} \Rightarrow 7^{4} \]

Here are some questions (As an extra Challenge Try doing these mentally)

1) \( 7637^{342} \mod 10 \) \hspace{1cm} [9]
2) \( 5639^{4298} \mod 10 \) \hspace{1cm} [1]
3) \( 8363^{4535} \mod 10 \) \hspace{1cm} [7]
4) \( 7854594^{4523454} \mod 10 \) \hspace{1cm} [6]
5) \( 56782^{93554350} \mod 10 \) \hspace{1cm} [4]

You can check these answers at the site-

http://www.wolframalpha.com/input/?i=525352%5E4234324%2Bmod%2B100
FINDING LAST TWO DIGITS OF ANY NUMBER TO ANY POWER

We are going to use some VM sutras to solve these types of problems. The Sutras that were used in the last be used are-

वेष्टनम्
Vestanam

Osculation

उर्ध्वार्द्धीयाभ्यामं
Urdhva Tiryagbhyam

Vertically and Crosswise

शिष्यते शेषसंजः
Sisyate Sesamjnah

The remainder remains constant

अन्त्ययोरेव
Antyayoreva

Only the last terms

एकाधिकेन पूर्वेण
ekādhikena pūrvena

By one more than the one before
Now that we are familiar with finding the last digit of powers, we are ready to tackle the next challenge - Finding the last two digits (LTD).

It is divided into two cases to increase the simplicity and efficiency of the VM Sutras. One is for odd number and one is for Even numbers. Dealing with these two cases will allow us to find the last two digits of any number to any power. We will also see a method that will give us our UD and Simultaneously give our Second last digit.

1) ODD CASE

When the base is odd, we follow the same steps as before but instead of retaining the last digit only, we retain the last two digits. i.e. in S1 we take the last two digits. We learnt in the previous topic of finding UD that we are concerned only with the last two digits of powers (Antyayoreva Sutra). So now we can write our S1 as the Base with last two digits and Power with the last two digits.

Eg -- $431^{324}$  \[\Rightarrow\]  S1 --- $31^{24}$

$978977^{432423}$  \[\Rightarrow\]  S1—$77^{23}$

Now we can look back into the table we made in finding the last digit

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If we look into the patterns of the odd numbers we see that 1 appears in every cycle (except for 5). This repetition of 1 in every cycle will help us find the LTD of any odd number easily.
We know that

\[(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{(1)}\]

\[= 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \cdots ,\]

Let's see how we can use this Interesting formula.

Let's say we want to find the LTD of \(71^{44}\)
From the last article we can say that the UD is 1.
We can write \(71^{44}\) as \((70+1)^{44}\)
So, the expansion becomes

\[(1+70)^{44} = 1 + 44 \times 70 + 44(43) \times 70^2 / 2 + \cdots \]

Only the first two terms are of our concern to find the LTD.
Thus we see the first two terms of the expansion are

\[1 + 44 \times 70 = 3080 + 1 = 81\]

So the LTD of \(71^{44}\) is 81.
In simple terms we can find the Last two terms of a number ending with 1 as follows-
We know that UD of a number ending with 1 is always one (Cycle 1,1,1,1)

\(71^{44}\)

\[\text{POWER} \quad \text{44} \quad \text{BASE} \quad 71 \quad \text{We do } 7 \times 4 = 28 \quad \text{LTD} \quad 81\]
Similarly, $31^{6346}$

**Exercise**

Find LTD’s Of the following-
1) $51^{424}$ (Remember that for the LTD $51$ or $4234351$ does not matter.)
2) $31^{567}$
3) $41^{424}$

Now let’s take some solid examples on how to extend this to any odd number.

1) **Find LTD of $764253^{472744}$**

We can find S1 as illustrated in the last article as

**S1** - $13^{44}$

**S2** —

We have seen in the table of powers that $3^4$ ends with a 1. So we try to write it in such a way that the base can be converted into the form which ends with 1. Here’s what we do -
We know that – $3^4$ ends with a 1.

So,

$13^{44} = 13^{4*11}$

We have to manually find the LTD of $13^4$ to find the LTD of $13^{4*11}$.
LTD of $13^4$ are –61.
So now we just have to find out the LTD of $61^{11}$ which we already learnt.
So the LTD of 764253^{472744} are 61.

2) Find $4779^{45} \mod 100$.

\begin{align*}
S_1 &= 79^{45} \\
S_2 &=
\text{Unlike our last question the power is not divisible by 4. What we do now is break it into two parts – one which is divisible by 4 and the other which is not.}
\end{align*}

\begin{align*}
79^{45} &= 79^{4(11)} \cdot 79^1 \\
So, what we did was we divided the power by 4. \\
45/4 &= 11 \text{ R}1
\end{align*}

\begin{align*}
\text{We put 11*4 (Quotient *4) on one side and the remainder part (R 1) on the other side.} \\
\text{Solving the Quotient part (Q)}
\end{align*}

\begin{align*}
\text{LTD of } 79^{4(11)} \\
\text{LTD of } 79^4 \text{ is ---81.} \\
\text{LTD of } 81^{11} \text{ are 81.}
\end{align*}

\begin{align*}
\text{Now solving Remainder Part (R)} \\
\text{LTD of } 79^1 \text{ is 79}
\end{align*}

\begin{align*}
\text{Our final answer Will be } (Q) \cdot (R) = 81 \cdot 79 = -- 99
\end{align*}
So \(4779^{45}\) mod 100 is 99.

Let’s do some more examples to understand the work better.

3) Find \(843479^{2343}\) mod 100

\[\text{S1} - 79^{43}\]
\[\text{S2} - \]

Just like we did in the last Question,
\[
79^{43} = 79^{4(10)} \times 79^{3}
\]
\[
\text{(Q)} \quad \text{(R)}
\]

\[
\text{Q)} \ 	ext{LTD of } 79^{4(10)}
\]
LTD of \(79^4\) are 81
Now LTD of \(81^{10}\) are 01.==(Q)

\[
\text{R) LTD of } 79^{3}
\]
(Simplifying such big calculations is given in the end of this chapter)

LTD of \(79^3\) are ---39===(R)
Final answer = Q * R = 01*39 = 39

Thus \(843479^{2343}\) mod 100 is 39.
4) Find LTD of $4324234312342$

\begin{align*}
S_1 & \quad 23^{42} \\
S_2 & \quad 23^{42} = 23^{40} \times 23^2 \\
& \quad \begin{array}{c}
Q \\
R
\end{array}
\end{align*}

\[ Q = 23^{4(10)} \]
LTD of $23^4$ are 41
LTD of $41^{10} \rightarrow 01$

\[ R \]
LTD of $23^2 \rightarrow 29$

Final Answer $= Q \times R$
$= 01 \times 29$
$= 29$

LTD of $4324234312342$ are 29.

5) Find last two digits of $2367^{434}$

\begin{align*}
S_1 & \quad 67^{34} \\
S_2 & \quad 67^{34} = 67^{4(8)} \times 67^2 \\
& \quad \begin{array}{c}
Q \\
R
\end{array}
\end{align*}

\[ Q \]
LTD of $67^4$ is --01
LTD of $01^8$ is ---01

\[ R \]
LTD of $67^2 \rightarrow --29$
Final LTD = Q * R
= 01 * 29
= 29

6) What is $4129^{42424}$ mod 100?

S1: $29^{24}$

S2: $29^{4(6)}$
No adjustment is to be done because it is divisible by 4.

LTD of $29^4$ is 81.

LTD of $81^6$

Thus $4129^{42424}$ mod 100 is 81.

7) What is $43245^{4234435}$
I didn’t tell what to do with 5’s at the end. Because it is the easiest of all. No working nothing, we simply write down 25-- That’s it. Here is the explanation--

$5^2 = 25$
$15^2 = 225$
$25^2 = 625$
$35^2 = -25$
This goes on forever….last digit is always 25
Many might be familiar with this trick, it was one of my first surprises in Vedic Mathematics.
The *Ekadhiken Purven* Sutra

\[ 25^2 = 2(2+1)/25 = 625 \]
\[ 35^2 = 3*4/25 = 1225 \]
\[ 505^2 = 50*51/25 = 2550/25 \]

Note that we are writing 25 every time at the end.

**SIMPLIFYING CALCULATIONS**

Here are some tricks to simplify the calculations.

- Firstly I would encourage everyone to learn the Urdhva tiryagbhyam sutra (Also known as Vertically And Crosswise) if not familiar. (See reference for books on the Topic.)

- Quick recap on the sutra to find last two digits

  \[(36)^2\]

  \[
  \begin{array}{ccc}
  3 & 6 & 6 \\
  3 & 6 & \\
  \hline
  9 & 6 & 6 \\
  3 & + & 3 \\
  \end{array}
  \]

  (Explanation - 1) starting from the end we write 6*6 = 36

  write 6 and carry 3

  2) now write 3*6 + 3*6 = 36 take 6 and carry over 3

  3) we don’t have to do any more to find last two digits we add 6 and 3 together to get 96 to find the total answer we do 3*3

Similarly,

Find LTD of \(43^4\)

What we do in this is first find the LTD of \(43^2\) and then again Square it to find the LTD of the given question.

Working-

\[
\begin{array}{c}
4 & 3 \\
4 & 3 \\
4 & 9 \\
2 & 1 \\
\hline
7 & + & 8 \\
\hline
\end{array}
\]

LTD of \(43^4\) is 01
- Those who are familiar with VM can also use Duplex Method to find the last two digits.

**SUMMARY (ODD CASE)**

- **S1** is the LTD of Base and LTD of Power.

- **In S2**
  - We first divide the Power by 4
  - We then Adjust the Base so that 1 comes at the end.
  - If needed we break into two parts – (Q) and (R)
  - We solve Q and R separately.
  - Final Answer = Q * R

- We use *Vertically and Crosswise Sutra* to Simplify calculations.

<table>
<thead>
<tr>
<th>Find LTD of the Following</th>
<th>Answers</th>
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<tr>
<td>1) 423949267^345245234</td>
<td>[29]</td>
</tr>
<tr>
<td>2) 543475723^546456456</td>
<td>[61]</td>
</tr>
<tr>
<td>3) 6563456456^6535634563</td>
<td>[63]</td>
</tr>
<tr>
<td>4) 6345^5345234</td>
<td>[25]</td>
</tr>
</tbody>
</table>
2) **EVEN CASE**

This is much easier than the odd case. Here are some things that we need to know --

\[
2^{10} = 1024 \\
2^{20} = 1048576
\]

For some reason I am yet to find out.

(Match the given results with the above data) ----

\[
2^{10\text{ (even)}} = 76 \quad \text{(Rule 1)} \\
2^{10\text{ (odd)}} = -24 \quad \text{(Rule 2)}
\]

Let's take some examples to illustrate the usage of the above two properties.

1) **Find \( 5464^{43554} \mod 100 \)**

Just like the Odd case our S1 will have no change.

\[
\begin{align*}
\text{S1} & : \quad 64^{54} \\
\text{S2} & : \quad \text{The change in S2 is that instead of breaking it and adjusting it like in the odd case,} \\
& \quad \text{We change the base to the power of 2.} \\
& \quad \text{We know that} \quad 64 = 8^2 = 2^{3(2)} = 2^6 \\
& \quad \text{So S1 becomes} \quad 2^6(54) = 2^{324} \\
& \quad \text{Now our new S1 becomes} \quad 2^{24} \\
& \quad \text{Our power is} \quad 2^{24}, \text{since} \quad 2 \text{ is even, it follows our} \quad \text{Rule 1} \\
& \quad \text{Final Answer will be} \quad 76 \times 2^{4} \\
& \quad = -16
\end{align*}
\]

Thus the LTD is 16.
2) Find \( 432744316^{547329} \mod 100 \)

\( S1 - 16^{29} \)

Now since \( 16 = 2^4 \)

\( S1 - 2^{4(29)} = 2^{116} \)

Or \( S1 - 2^{16} \)

\( S2 - \)

In \( 2^{16} \) power is 16. 1 is odd so Rule 2 follows.

Final answer = \( 24 \cdot 2^6 \)

\( = 24 \cdot 64 \)

\( = 1536 \)

LTD is 36.

3) Find LTD of \( 32434232^{534534532} \)

Just like before,

\( S1 - 32^{32} \)

\( \quad - 2^{5(32)} \quad ---- \quad 2^{160} \)

\( \quad ---- \quad 2^{60} \)

\( S2 - \)

The power is 60. 6 is even, so Rule 1 follows.

Final Answer = \( 76 \cdot 2^0 \) (from 60)

LTD is 76.

4) Find LTD of \( 3423402^{45674232} \)

\( S1 - 02^{32} \)

\( S2 - \)

In 32, 3 is odd. So Rule 2 applies.

Final Answer = \( 24 \cdot 2^2 \)

\( = 24 \cdot 4 \)

\( = 96 \)

LTD is 96.
SUMMARY (EVEN CASE)

\[
\begin{align*}
2^{10}\text{(even)} & = \text{---76 (Rule 1)} \\
2^{10}\text{(odd)} & = \text{---24 (Rule 2)}
\end{align*}
\]

- In S1 we break down the base into Powers of 2 and solve.
- In S2 we look at the first digit of power and apply Rule 1 or 2 accordingly.
- In the given method we find the last digit as well as second last digit simultaneously.
So first method can be used as an extra proof of the correctness of the answer.

-Have a look at the following pattern

| 76  * 02 | ------02 |
| 76  * 04 | ------04 |
| 76  * 08 | ------08 |
| 76  * 16 | ------16 |
| 76  * 32 | ------32 |

And so on, I don’t Know why it is like that, but it Works!!
So, wherever Rule 1 applies you can use this.

\[
76 \times 2^n = \text{----}2^n
\]

Exercise – Find LTD of the Following.

Answers.

1) 4234616\text{43534252} \text{ (Hint: 0 is Even)} \quad [56]
2) 32942904\text{5466345} \quad [24]
3) 48952902\text{34563456} \quad [36]
Mixed Problems

Now we are ready to take up mixed problems where both even and odd cases apply.

Q) LTD of 134123443556^435245.

S1 - 56^45

S2 -
We divide S1 in two parts, One with Even and one with Odd.
56 = 2 * 28 = 2^2 * 14 = 2^3 * 7 (Keep dividing by 2 until you get and odd no.)

56^45 = 2^{3(45)} * 07^{45}

[EVEN]       [ODD]
Solving Even and Odd Separately,

[EVEN] = 2^{135}
LTD is 68
[ODD] = 7^{45}
Easy enough, LTD is 07

Final LTD = [EVEN] * [ODD] = 68 * 07 = --- 76

Q) LTD of 44141224^4234234234
S1 = 24^34

S2 = 2^{3(34)} * 3^{34}

[E]       [O]

[E] = LTD is 4
[O] = LTD is 69
FINAL LTD = [E] * [O] = 4 * 69 = ----76
CONCLUSION

We have learnt how to find the last and the last two digits of any exponential number. The procedure we saw used Vedic Mathematics and it is well capable of outdoing various Theorems on the issue such as Binomial Theorem and Expansion, Combinatorics, Fermat's Little Theorem and others.

REFERENCES

-Vedic Mathematics - Swami Bharati Krishna Tirthaji Maharaja;

-Urdhva Tiryakbhyam or Vertically and Crosswise - Kenneth Williams, A.P.Nicholas

Site which finds the last two digits

--http://www.wolframalpha.com/input/?i=525352%5E4234324+mod+100